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# Generalized spectral tests for serial dependence

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**Summary.** Two tests for serial dependence are proposed using a generalized spectral theory in combination with the empirical distribution function. The tests are generalizations of the Cramér–von Mises and Kolmogorov–Smirnov tests based on the standardized spectral distribution function. They do not involve the choice of a lag order, and they are consistent against all types of pairwise serial dependence, including those with zero autocorrelation. They also require no moment condition and are distribution free under serial independence. A simulation study compares the finite sample performances of the new tests and some closely related tests. The asymptotic distribution theory works well in finite samples. The generalized Cramér–von Mises test has good power against a variety of dependent alternatives and dominates the generalized Kolmogorov–Smirnov test. A local power analysis explains some important stylized facts on the power of the tests based on the empirical distribution function.

*Keywords*: Cramér–von Mises criterion; Distribution-free tests; Empirical distribution function; Generalized spectrum; Kolmogorov–Smirnov criterion; Multiparameter empirical process; Non-linear time series; Weak convergence

## 1. Introduction

Let $\{X_t \in \mathbb{R}\}_{t=-\infty}^{\infty}$ be a stationary process with marginal distribution function $G(x) := P(X_t \leqslant x)$ and pairwise distribution function $F_j(x, y) := P(X_t \leqslant x, X_{t-j} \leqslant y)$, where $(x, y) \in \mathbb{R}^2$ and $j = 0, \pm 1, \ldots$. Suppose that we have a random sample $\{X_t\}_{t=1}^{n}$ of size $n$, and we are interested in testing the null hypothesis $\mathbb{H}_0$ that $\{X_t\}$ is independently and identically distributed (IID).

The dependence of $\{X_t\}$ often is characterized by the standardized spectral density

$$h(\omega) := (2\pi)^{-1} \sum_{j=-\infty}^{\infty} \rho(j) \exp(-\mathrm{i} j\omega), \qquad \omega \in [-\pi, \pi], \quad \mathrm{i} := \sqrt{-1}, \tag{1.1}$$

where $\rho(j) := \operatorname{corr}(X_t, X_{t-j})$, or by the standardized spectral distribution function

$$H(\lambda) := 2 \int_0^{\lambda\pi} h(\omega)\, \mathrm{d}\omega = \lambda + 2 \sum_{j=1}^{\infty} \rho(j) \frac{\sin(j\pi\lambda)}{j\pi}, \qquad \lambda \in [0, 1]. \tag{1.2}$$

Under hypothesis $\mathbb{H}_0$, $H(\lambda)$ becomes $H_0(\lambda) := \lambda$. To test $\mathbb{H}_0$, Anderson (1993) compared $H_0(\lambda)$ with

$$\hat{H}(\lambda) := \lambda + 2 \sum_{j=1}^{n-1} \hat{\rho}(j) \frac{\sin(j\pi\lambda)}{j\pi}, \tag{1.3}$$

where $\hat{\rho}(j)$ is the sample autocorrelation function, via the Cramér–von Mises criterion

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$$T_{\mathrm{CM}} := \frac{n}{2} \int_0^1 \{\hat{H}(\lambda) - H_0(\lambda)\}^2 \,\mathrm{d}\lambda = n \sum_{j=1}^{n-1} \frac{\hat{\rho}^2(j)}{(j\pi)^2} \tag{1.4}$$

and the Kolmogorov–Smirnov criterion

$$T_{\mathrm{KS}} := \left(\frac{n}{2}\right)^{1/2} \sup_{\lambda \in [0,1]} |\hat{H}(\lambda) - H_0(\lambda)| = \sup_{\lambda \in [0,1]} \left| n^{1/2} \sum_{j=1}^{n-1} \hat{\rho}(j) \frac{\sqrt{2}\, \sin(j\pi\lambda)}{j\pi} \right|. \tag{1.5}$$

The tests using $T_{\mathrm{CM}}$ and $T_{\mathrm{KS}}$ are consistent against all types of autocorrelation, but they have no power against alternatives with zero autocorrelation. Examples include autoregressive conditional heteroscedastic (ARCH), bilinear, non-linear moving average, threshold autoregressive processes and iterative logistic maps (see Hall and Wolff (1995)). Moreover, $E(X_t^2) < \infty$ is required for $T_{\mathrm{CM}}$ and $T_{\mathrm{KS}}$. This rules out time series with infinite variance, as is often observed in high frequency economic and financial data (e.g. Mandelbrot (1967)).

To test generic serial dependence and to avoid the moment condition, we shall generalize Anderson's (1993) approach via the dependence measure

$$\rho_j^*(x, y) := F_j(x, y) - G(x)\,G(y), \qquad (x, y) \in \mathbb{R}^2, \quad j = 0, \pm 1, \ldots. \tag{1.6}$$

Because $\rho_j^*(x, y) = 0 \ \forall (x, y) \in \mathbb{R}^2$ if and only if $X_t$ and $X_{t-j}$ are independent, $\rho_j^*(x, y)$ can capture all types of pairwise dependence, including those with zero autocorrelation.

The idea of using measure (1.6) to test independence dates back to Hoeffding (1948), who used an analogue of it to test independence between two IID random variables. Blum *et al.* (1961) used the empirical distribution function (EDF) to test independence between the components of an IID random vector via the Cramér–von Mises and Kolmogorov–Smirnov criteria. See also Deheuvels (1981) and Carlstein (1988).

Testing for hypothesis $\mathbb{H}_0$ via measure (1.6) is more complicated because the lag order $j$ is involved. Skaug and Tjøstheim (1993a) were the first to test pairwise serial dependence up to order $p$ by using measure (1.6). Hong (1998) proposed a complementary test. Delgado (1996) extended the Cramér–von Mises test of Blum *et al.* (1961) to detect joint dependence of $\{X_t, \ldots, X_{t-p}\}$. There are other time domain tests for serial dependence (e.g. Brock *et al.* (1996), Chan and Tran (1992), Hjellvik and Tjøstheim (1996), Pinkse (1998), Robinson (1991), Scargle (1981), Skaug and Tjøstheim (1993b, 1996) and Wolff (1994)). See Tjøstheim (1996) for an excellent survey.

Almost all the existing tests deal with serial dependence of a finite order. This is unsatisfactory from a theoretical point of view, because the actual dependence may be of a higher order. In practice, as Skaug and Tjøstheim (1993a, 1996) observed, the power of these tests heavily depends on the chosen lag order. Often, maximal power is achieved by using the correct lag order of the alternative. However, prior information on the dependence structure is usually not available. For such alternatives as fractionally integrated processes, it is difficult to choose a lag order to maximize power even if we knew the alternative. Practitioners often have to try several lags. It is quite common that some lags give significant statistics whereas others do not. Thus, it is delicate to reach a decisive conclusion.

The key idea of this paper is to synthesize the approaches of Anderson (1993) and Skaug and Tjøstheim (1993a). The new tests do not involve the choice of a lag order and are consistent against all types of pairwise dependence. They require no moment condition and have accurate sizes in finite samples. It should be emphasized, however, that the technical convenience of the lag independence does not necessarily give the best power. Indeed, the new tests should be viewed as not competing but as a complement to such tests as those of Skaug and Tjøstheim (1993a) and Hong (1998). In subsequent sections, we introduce the approach and test statistics, establish the asymptotic distribution and power theory, and compare the tests with some closely related tests in finite samples. Mathematical proofs are available from the author. Throughout the paper, all convergences are taken as $n \to \infty$.

## 2. A generalized spectral approach

Because $\rho_j^*(x, y) = 0 \ \forall (x, y) \in \mathbb{R}^2$ implies $\rho_{-j}^*(x, y) = 0 \ \forall (x, y) \in \mathbb{R}^2$, and vice versa, to test hypothesis $\mathbb{H}_0$ it suffices to check whether $\rho_j^*(x, y) = 0$ for all $j > 0$. We thus base our tests on

$$\rho_j(x, y) := \rho_{|j|}^*(x, y), \qquad j = 0, \ \pm 1, \ldots. \tag{2.1}$$

This greatly simplifies the analysis. Observing that $\rho_j(x, y)$ is the covariance between $\mathbf{1}(X_t \leqslant x)$ and $\mathbf{1}(X_{t-|j|} \leqslant y)$, where $\mathbf{1}(\cdot)$ is the indicator function, we consider its Fourier transform

$$h(\omega, x, y) := (2\pi)^{-1} \sum_{j=-\infty}^{\infty} \rho_j(x, y) \exp(-\mathrm{i} j \omega), \qquad \omega \in [-\pi, \pi]. \tag{2.2}$$

This exists if

$$\sup_{(x,y)\in\mathbb{R}^2} \left\{ \sum_{j=-\infty}^{\infty} |\rho_j(x, y)| \right\} < \infty,$$

which holds under a proper mixing condition. It is a generalization of the standardized spectral density $h(\omega)$ in expression (1.1). Similarly,

$$H(\lambda, x, y) := 2 \int_0^{\lambda\pi} h(\omega, x, y)\, \mathrm{d}\omega = \rho_0(x, y)\lambda + 2 \sum_{j=1}^{\infty} \rho_j(x, y) \frac{\sin(j\pi\lambda)}{j\pi}, \qquad \lambda \in [0, 1], \tag{2.3}$$

is a generalization of the standardized spectral distribution function $H(\lambda)$ in expression (1.2). The functions (2.1)–(2.3) can capture all types of pairwise dependence and require no moment condition, which are not attainable by $\rho(j)$, $h(\omega)$ or $H(\lambda)$. They also differ from higher order spectra (Brillinger and Rosenblatt, 1967a, b; Subba Rao and Gabr, 1984), which can characterize many types of non-linearity but may still miss some important ones. The bispectrum, for example, easily misses ARCH processes with zero third-order cumulants. Moreover, higher order moment conditions are required for higher order spectra.

The spectral approach provides a natural tool to test hypothesis $\mathbb{H}_0$. Under $\mathbb{H}_0$, $H(\lambda, x, y)$ becomes

$$H_0(\lambda, x, y) := \rho_0(x, y)\lambda. \tag{2.4}$$

This is analogous to the flat spectrum $H_0(\lambda) = \lambda$. Define the empirical measure

$$\hat{\rho}_j(x, y) := \hat{F}_j(x, y) - \hat{F}_j(x, \infty)\, \hat{F}_j(\infty, y), \qquad j = 1, \ldots, n-1, \tag{2.5}$$

where

$$\hat{F}_j(x, y) := (n-j)^{-1} \sum_{t=j+1}^{n} \mathbf{1}(X_t \leqslant x)\, \mathbf{1}(X_{t-j} \leqslant y)$$

is an unbiased EDF of $(X_t, X_{t-j})$. Two plausible estimators for $H(\lambda, x, y)$ and $H_0(\lambda, x, y)$ are

$$\hat{H}(\lambda, x, y) := \hat{\rho}_0(x, y)\lambda + 2\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)^{1/2}\hat{\rho}_j(x, y)\frac{\sin(j\pi\lambda)}{j\pi}, \tag{2.6}$$

$$\hat{H}_0(\lambda, x, y) := \hat{\rho}_0(x, y)\lambda. \tag{2.7}$$

In expression (2.6), $\hat{\rho}_j(x, y)$ is naturally weighted down for higher order lags. The factor $(1 - j/n)^{1/2}$ is a small sample correction, further discounting higher order lags.

Put $\hat{G}(x) := n^{-1}\sum_{t=1}^{n}\mathbf{1}(X_t \leqslant x)$. To test for hypothesis $\mathbb{H}_0$, we can use the $L_2$-norm

$$T_{\mathrm{GCM}} := \frac{n}{2}\int_0^1\int_{\mathbb{R}^2}\{\hat{H}(\lambda, x, y) - \hat{H}_0(\lambda, x, y)\}^2\,\mathrm{d}\hat{G}(x)\,\mathrm{d}\hat{G}(y)\,\mathrm{d}\lambda = \sum_{j=1}^{n-1}\frac{(n-j)\,\hat{\sigma}^2(j)}{(j\pi)^2}, \tag{2.8}$$

where

$$\hat{\sigma}^2(j) := \int_{\mathbb{R}^2}\hat{\rho}_j^2(x, y)\,\mathrm{d}\hat{G}(x)\,\mathrm{d}\hat{G}(y) = n^{-2}\sum_{t=1}^{n}\sum_{s=1}^{n}\hat{\rho}_j^2(X_t, X_s).$$

Note that $\hat{\sigma}^2(j)$ differs from the empirical Hoeffding (1948) dependence measure

$$\tilde{\sigma}^2(j) := \int_{\mathbb{R}^2}\hat{\rho}_j^2(x, y)\,\mathrm{d}\hat{F}_j(x, y) = (n-j)^{-1}\sum_{t=j+1}^{n}\hat{\rho}_j^2(X_t, X_{t-j}), \tag{2.9}$$

which is used in Skaug and Tjøstheim (1993a) and Hong (1998). Both $\hat{\sigma}^2(j)$ and $\tilde{\sigma}^2(j)$ converge to 0 under hypothesis $\mathbb{H}_0$, but generally to different limits under the alternative.

Another test statistic is the supremum norm

$$\begin{aligned} T_{\mathrm{GKS}} &:= \left(\frac{n}{2}\right)^{1/2}\sup_{(x,y)\in\mathbb{R}^2}\sup_{\lambda\in[0,1]}|\hat{H}(\lambda, x, y) - \hat{H}_0(\lambda, x, y)| \\ &= \max_{1\leqslant t,s\leqslant n}\sup_{\lambda\in[0,1]}\left|\sum_{j=1}^{n-1}(n-j)^{1/2}\,\hat{\rho}_j(X_t, X_s)\frac{\sqrt{2}\,\sin(j\pi\lambda)}{j\pi}\right|. \end{aligned} \tag{2.10}$$

The statistics $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$ are generalizations of $T_{\mathrm{CM}}$ and $T_{\mathrm{KS}}$ in expressions (1.4) and (1.5). Because $T_{\mathrm{GCM}}$ is invariant under any order preserving transformation, and $T_{\mathrm{GKS}}$ is invariant under any continuous monotonic transformation, they are distribution free under hypothesis $\mathbb{H}_0$. This is appealing because $X_t$ and $X_{t-j}$ are independent if and only if $g(X_t)$ and $g(X_{t-j})$ are independent for any continuous monotonic function $g$: $\mathbb{R} \to \mathbb{R}$ (e.g. Skaug and Tjøstheim (1996)). The test statistics of Skaug and Tjøstheim (1993a), Delgado (1996) and Hong (1998) are distribution free under $\mathbb{H}_0$ as well. Note that the distribution-free property holds only when $X_t$ is a continuous variable (see Skaug and Tjøstheim (1993a)).

## 3. Asymptotic null distribution

In this section, we first establish a functional central limit theorem for a multiparameter empirical process. The null limit distributions of the new tests are then derived as the limit distributions of some continuous functionals of the multiparameter empirical process.

*Assumption 1.* $\{X_t\}$ is a stationary process with continuous marginal distribution $G(\cdot)$.

Put $U_t := G(X_t)$, which follows a uniform marginal distribution on [0, 1], and put $\gamma := (\lambda, u, v) \in \mathbb{I} := [0, 1]^3$, with metric $\theta(\gamma_1, \gamma_2) := |\lambda_1 - \lambda_2| + |u_1 - u_2| + |v_1 - v_2|$. Define

$$\hat{H}^U(\gamma) := \hat{\rho}_0^U(u, v)\lambda + 2\sum_{j=1}^{n-1}\left(1 - \frac{j}{n}\right)^{1/2}\hat{\rho}_j^U(u, v)\frac{\sin(j\pi\lambda)}{j\pi}, \tag{3.1}$$

$$\hat{H}_0^U(\gamma) := \hat{\rho}_0^U(u, v)\lambda, \tag{3.2}$$

where $\hat{\rho}_j^U(u, v) := \hat{F}_j^U(u, v) - \hat{F}_j^U(u, 1)\,\hat{F}_j^U(1, v)$ and $\hat{F}_j^U(u, v)$ is the EDF of $(U_t, U_{t-j})$, defined in the same way as $\hat{F}_j(x, y)$. Given assumption 1, the inverse function $G^{-1}$ of $G(\cdot)$ exists and $\hat{\rho}_j^U(u, v) = \hat{\rho}_j\{G^{-1}(u), G^{-1}(v)\}$, whence, with $\hat{G}^U(u) := n^{-1}\,\Sigma_{t=1}^n\,\mathbf{1}(U_t \leqslant u)$, we have

$$T_{\mathrm{GCM}} = (n/2)\int_{\mathbb{I}}\{\hat{H}^U(\gamma) - \hat{H}_0^U(\gamma)\}^2\,\mathrm{d}\hat{G}^U(u)\,\mathrm{d}\hat{G}^U(v)\,\mathrm{d}\lambda, \tag{3.3}$$

$$T_{\mathrm{GKS}} = (n/2)^{1/2}\sup_{\gamma\in\mathbb{I}}|\hat{H}^U(\gamma) - \hat{H}_0^U(\gamma)|. \tag{3.4}$$

We shall first show that, under hypothesis $\mathbb{H}_0$, $(n/2)^{1/2}\{\hat{H}^U(\gamma) - \hat{H}_0^U(\gamma)\}$ can be approximated by

$$Y_n(\gamma) := \sum_{j=1}^{n-1}(n-j)^{1/2}\,\hat{r}_j(u, v)\frac{\sqrt{2}\sin(j\pi\lambda)}{j\pi}, \tag{3.5}$$

where

$$\hat{r}_j(u, v) := (n-j)^{-1}\sum_{t=j+1}^{n}\{\mathbf{1}(U_t \leqslant u) - u\}\{\mathbf{1}(U_{t-j} \leqslant v) - v\}.$$

*Lemma 1*. Under assumption 1 and hypothesis $\mathbb{H}_0$,

$$\sup_{\gamma\in\mathbb{I}}\left|\left(\frac{n}{2}\right)^{1/2}\{\hat{H}^U(\gamma) - \hat{H}_0^U(\gamma)\} - Y_n(\gamma)\right| \xrightarrow{\mathrm{p}} 0.$$

Lemma 1 and equations (3.3) and (3.4) imply

$$T_{\mathrm{GCM}} - \int_{\mathbb{I}} Y_n^2(\gamma)\,\mathrm{d}\hat{G}^U(u)\,\mathrm{d}\hat{G}^U(v)\,\mathrm{d}\lambda \xrightarrow{\mathrm{p}} 0, \tag{3.6}$$

$$T_{\mathrm{GKS}} - \sup_{\gamma\in\mathbb{I}}|Y_n(\gamma)| \xrightarrow{\mathrm{p}} 0. \tag{3.7}$$

Thus, we can focus on the multiparameter stochastic process $Y_n(\gamma)$. Let $D(\mathbb{I})$ denote the space of real-valued functions on $\mathbb{I}$ that are right continuous with left-hand limits existing at each point of $\mathbb{I}$. We equip $D(\mathbb{I})$ with the Skorohod metric. Then $Y_n(\gamma)$ is a random element in $D(\mathbb{I})$. Let $Z(\gamma)$ be a Gaussian process with mean 0 and covariance kernel

$$E\{Z(\gamma)\,Z(\gamma')\} := (u \wedge u' - uu')(v \wedge v' - vv')(\lambda \wedge \lambda' - \lambda\lambda'), \tag{3.8}$$

where $u \wedge v := \min(u, v)$. We may call $Z(\gamma)$ a three-parameter separable Brownian bridge on $\mathbb{I}$. By showing that the finite dimensional distribution of $Y_n(\gamma)$ converges to that of $Z(\gamma)$ and that $Y_n(\gamma)$ is stochastically equicontinuous on the metric space $(\mathbb{I}, \theta)$, we obtain the functional central limit theorem for $Y_n(\gamma)$, one of the key results in this paper.

*Theorem 1*. Under assumption 1 and hypothesis $\mathbb{H}_0$, $Y_n(\gamma)$ converges weakly to $Z(\gamma)$ on $D(\mathbb{I})$.

This can be used to construct many tests for serial dependence using various continuous divergence measures, the $L_2$-norm and the supremum norm being just two examples.

To obtain the limit distribution of $T_{\mathrm{GCM}}$, we show that replacing the empirical measure $\hat{G}^U(\cdot)$ in equation (3.6) by its theoretical counterpart does not affect the limit distribution of $T_{\mathrm{GCM}}$.

*Theorem 2*. Under assumption 1 and hypothesis $\mathbb{H}_0$,

$$\int_{\mathbb{I}} Y_n^2(\gamma)\,\mathrm{d}\hat{G}^U(u)\,\mathrm{d}\hat{G}^U(v)\,\mathrm{d}\lambda - \int_{\mathbb{I}} Y_n^2(\gamma)\,\mathrm{d}\gamma \xrightarrow{\mathrm{p}} 0.$$

This, along with theorem 1, equations (3.6) and (3.7) and the continuous mapping theorem (see Pollard (1990)), yields the limit distributions of the tests proposed.

*Theorem 3*. Under assumption 1 and hypothesis $\mathbb{H}_0$,

$$T_{\mathrm{GCM}} \xrightarrow{\mathrm{d}} \int_{\mathbb{I}} Z^2(\gamma)\,\mathrm{d}\gamma$$

and

$$T_{\mathrm{GKS}} \xrightarrow{\mathrm{d}} \sup_{\gamma\in\mathbb{I}} |Z(\gamma)|.$$

To study these limit distributions, let $\lambda_j > 0$ be an eigenvalue and $\psi_j$: $[0, 1] \rightarrow \mathbb{R}$ the corresponding normalized eigenfunction of the integral equation

$$\psi(u) = \lambda \int_0^1 q(u, v)\,\psi(v)\,\mathrm{d}v$$

for some positive definite function $q$: $[0, 1]^2 \rightarrow \mathbb{R}$. We can express

$$q(u, v) = \sum_{j=1}^{\infty} \lambda_j^{-1}\,\psi_j(u)\,\psi_j(v),$$

where the series converges absolutely and uniformly on $[0, 1]^2$. For $q(u, v) = u \wedge v - uv$, we have $\lambda_j = (j\pi)^{-2}$ and $\psi_j(u) = \sqrt{2}\,\sin(j\pi u)$, as is well known. It follows that

$$E\,Z(\gamma)\,Z(\gamma') = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\sqrt{2}\sin(j\pi u)}{j\pi}\,\frac{\sqrt{2}\,\sin(k\pi v)}{k\pi}\,\frac{\sqrt{2}\,\sin(l\pi\lambda)}{l\pi}\,\frac{\sqrt{2}\,\sin(j\pi u')}{j\pi}$$
$$\times \frac{\sqrt{2}\,\sin(k\pi v')}{k\pi}\,\frac{\sqrt{2}\,\sin(l\pi\lambda')}{l\pi}.$$

Because $Z(\gamma)$ is Gaussian with mean 0, it has the Karhunen–Loève representation

$$Z(\gamma) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\sqrt{2}\,\sin(j\pi u)}{j\pi}\,\frac{\sqrt{2}\,\sin(k\pi v)}{k\pi}\,\frac{\sqrt{2}\,\sin(l\pi\lambda)}{l\pi}\,Z_{jkl},$$

where the $Z_{jkl}$ are independent $N(0, 1)$ variables. Thus, the limit distribution of $T_{\mathrm{GCM}}$

$$\int_{\mathbb{I}} Z^2(\gamma)\,\mathrm{d}\gamma = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(j\pi)^2}\,\frac{1}{(k\pi)^2}\,\frac{1}{(l\pi)^2}\,Z_{jkl}^2. \tag{3.9}$$

This is the same as the so-called Duque statistic, which arises in a different, multivariate, context (see Deheuvels (1981), page 110). In fact, equation (3.9) is not entirely unexpected in view of the earlier results in the present context (see Skaug and Tjøstheim (1993a), theorem 2).

The characteristic function of the limit random variable in equation (3.9) is

$$\varphi(\tau) := E\left\{\mathrm{i}\tau\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(j\pi)^2}\frac{1}{(k\pi)^2}\frac{1}{(l\pi)^2}Z_{jkl}^2\right\} = \prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\prod_{l=1}^{\infty}\left\{1-\frac{2\mathrm{i}\tau}{(j\pi)^2(k\pi)^2(l\pi)^2}\right\}^{-1/2}.$$

Its quantiles can be tabulated by numerical inversion of $\varphi(\tau)$, but an accurate tabulation seems difficult. Alternatively, we can simulate a truncated series

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{1}{(j\pi)^2}\frac{1}{(k\pi)^2}\frac{1}{(l\pi)^2}Z_{jkl}^2 \tag{3.10}$$

with a large $N$. Similarly, we can simulate the quantiles of $\sup_{\gamma\in\mathbb{I}}|Z(\gamma)|$ using

$$\sup_{\gamma\in\mathbb{I}}\left|\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{\sqrt{2}\,\sin(j\pi u)}{j\pi}\frac{\sqrt{2}\,\sin(k\pi v)}{k\pi}\frac{\sqrt{2}\,\sin(l\pi\lambda)}{l\pi}Z_{jkl}\right|. \tag{3.11}$$

In practice, the distribution-free property ensures that the exact null distributions of $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$ for any $n$ can be obtained by directly simulating the statistics $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$.

Hong (1999) considered an analogous generalized spectral approach based on the empirical characteristic function. There the limit distribution of the test statistic depends on the unknown distribution generating the data and thus cannot be tabulated. The method of analysis differs as well, because the indicator function is discontinuous whereas the complex-valued exponential function is analytic.

## 4. Asymptotic power

To state the consistency theorem, we impose the following condition.

*Assumption 2.*

(a) $\{X_t\}$ is a stationary mixing process with strong mixing coefficient $\alpha(j) \leqslant Cj^{-\nu}$ for some $\nu > 1$, where $C \in (0, \infty)$ does not depend on $j$;
(b) the joint distribution of $\{U_t, U_{t-j}\}$ has a continuous joint density bounded by $C$ on $[0, 1]^2$.

We first state the uniform convergence of $\hat{H}^U(\gamma)$ and $\hat{H}_0^U(\gamma)$.

*Lemma 2.* Under assumptions 1 and 2, $\sup_{\gamma\in\mathbb{I}}|\hat{H}^U(\gamma) - H^U(\gamma)| \to^{\mathrm{p}} 0$ and $\sup_{\gamma\in\mathbb{I}}|\hat{H}_0^U(\gamma) - H_0^U(\gamma)| \to^{\mathrm{p}} 0$, where $H^U(\gamma)$ and $H_0^U(\gamma)$ are defined in the same way as $H(\lambda, x, y)$ and $H_0(\lambda, x, y)$ in equations (2.3) and (2.4), with $U_t$ replacing $X_t$.

This ensures the consistency of $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$ against all types of pairwise dependence.

*Theorem 4.* Under assumptions 1 and 2,

$$(2/n)T_{\mathrm{GCM}} \xrightarrow{\mathrm{p}} \int_{\mathbb{I}}\{H^U(\gamma) - H_0^U(\gamma)\}^2\,\mathrm{d}\gamma$$

and

$$(2/n)^{1/2}T_{\mathrm{GKS}} \xrightarrow{\mathrm{p}} \sup_{\gamma\in\mathbb{I}}|H^U(\gamma) - H_0^U(\gamma)|.$$

Recall that $H^U(\gamma) = H_0^U(\gamma)$ if and only if $X_t$ and $X_{t-j}$ are independent for all $j > 0$. When

$\{X_t\}$ is not pairwise independent, $\Pr(T_{\mathrm{GCM}} > C_n) \to 1$ for any $\{C_n = o(n)\}$ and $\Pr(T_{\mathrm{GKS}} > C_n) \to 1$ for any $\{C_n = o(n^{1/2})\}$. Thus $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$ are consistent against all types of pairwise serial dependence. Most existing nonparametric tests do not have this property because they deal with serial dependence of a finite order. Of course, the consistency against all types of pairwise dependence is an asymptotic notion. Given any $n$, only $n-1$ lags can be checked. Also, $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$ cannot detect alternatives that are pairwise independent, although practical examples of such alternatives may not exist.

To gain additional insight, we consider a class of local alternatives for which the conditional density of $X_t$ given $\mathcal{F}_{t-1}$, the $\sigma$-field consisting of $X_s$, $s < t$, exists and is

$$\mathbb{H}_n\colon\ g_n(X_t|\mathcal{F}_{t-1}) = g(X_t)\{1 + n^{-1/2} f(X_t, X_{t-d}) + r_n(X_t, X_{t-d})\}, \tag{4.1}$$

where $g(\cdot) := G'(\cdot)$ is the marginal density of $X_t$ and $d$ is an arbitrary but fixed lag order.

*Assumption 3.*

(a) $\int_{-\infty}^{\infty} f(x, z)\,\mathrm{d}G(z) = \int_{-\infty}^{\infty} f(z, y)\,\mathrm{d}G(z) = 0 \ \ \forall(x, y) \in \mathbb{R}^2$ and $\int_{\mathbb{R}^2} |f(x, y)|^3\,\mathrm{d}G(x)\,\mathrm{d}G(y) < \infty$;
(b) $\int_{-\infty}^{\infty} r_n(x, z)\,\mathrm{d}G(z) = \int_{-\infty}^{\infty} r_n(z, y)\,\mathrm{d}G(z) = 0 \ \ \forall(x, y) \in \mathbb{R}^2$ and $\forall n > 0$, and $\int_{\mathbb{R}^2} |r_n(x, y)|^k\,\mathrm{d}G(x)\,\mathrm{d}G(y) = o(n^{-1})$ for $k = 2, 3$;
(c) $1 + n^{-1/2} f(x, y) + r_n(x, y) \geqslant 0\ \forall(x, y) \in \mathbb{R}^2$ and $\forall n > 0$.

Recalling that $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$, we define

$$\mu(\gamma) := \frac{\sqrt{2}\,\sin(d\pi\lambda)}{d\pi} \int_0^u \int_0^v f\{G^{-1}(\tilde{u}), G^{-1}(\tilde{v})\}\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}. \tag{4.2}$$

*Theorem 5.* Under assumption 3 and hypothesis $\mathbb{H}_n$,

$$T_{\mathrm{GCM}} \xrightarrow{\mathrm{d}} \int_{\mathbb{I}} |Z(\gamma) + \mu(\gamma)|^2\,\mathrm{d}\gamma$$

and

$$T_{\mathrm{GKS}} \xrightarrow{\mathrm{d}} \sup_{\gamma\in\mathbb{I}} |Z(\gamma) + \mu(\gamma)|.$$

Thus, $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$ have non-trivial power under hypothesis $\mathbb{H}_n$, which allows for any fixed but arbitrary lag order $d$ and converges to $\mathbb{H}_0$ at the parametric rate $n^{-1/2}$. Theorem 5 is useful in explaining some important stylized facts observed in the simulation below. We expect that the tests of Skaug and Tjøstheim (1993a) and Delgado (1996) also have power under $\mathbb{H}_n$ if the lag order used is larger than or equal to $d$.

## 5. Finite sample performance

### 5.1. Size

To assess the adequacy of the asymptotic theory in finite samples, we first generate 10000 realizations of $\{X_t\}_{t=1}^n$ from the uniform distribution on [0, 1] for $n = 10, 20, 30, 40, 50, 60, 80, 100$, using the GAUSS Windows NT/95 version 3.2.37 random number generator `RNDUS()`. We obtain 10000 statistics for $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$, where $T_{\mathrm{GKS}}$ is evaluated as

$$\max_{1\leqslant t,s\leqslant n}\ \max_{0\leqslant l\leqslant 100}\left|\sum_{j=1}^{n-1}\sqrt{2}(n-j)^{1/2}\,\hat{\rho}_j(X_t,\ X_s)\frac{\sin(j\pi l/100)}{j\pi}\right|.$$

Table 1 gives the resulting empirical critical values. The results show rather rapid convergence of the empirical critical values as $n$ increases. At each level, the variations in the critical values across different sample sizes are within 2.0% for $T_{\mathrm{GCM}}$ after $n \geqslant 50$, and within 2.8% for $T_{\mathrm{GKS}}$ after $n \geqslant 50$.

We now use the asymptotic theory and bootstrap to compare the sizes of $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$ with those of Anderson (1993), Skaug and Tjøstheim (1993a), Delgado (1996) and Hong (1998). We evaluate Anderson's (1993) $T_{\mathrm{KS}}$ as

$$\max_{0\leqslant l\leqslant 100}\left|n^{1/2}\sum_{j=1}^{n-1}\sqrt{2}\,\hat{\rho}(j)\frac{\sin(j\pi l/100)}{j\pi}\right|.$$

Skaug and Tjøstheim's (1993a) statistic $T_{\mathrm{ST1}(p)} := (n-1)\,\Sigma_{j=1}^{p}\,\tilde{\sigma}^2(j)$, where $\tilde{\sigma}^2(j)$ is as in expression (2.9). We include $T_{\mathrm{ST2}(p)} := \Sigma_{j=1}^{p}\,(n-j)\,\tilde{\sigma}^2(j)$, which is found in Hong (1998) to have better sizes. Under hypothesis $\mathbb{H}_0$, $T_{\mathrm{ST1}(p)}$ and $T_{\mathrm{ST2}(p)}$ converge in distribution to

$$\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}(j\pi)^{-2}(l\pi)^{-2}\,\chi_{jl}^2(p),$$

where the $\chi_{jl}^2(p)$ are independent $\chi_p^2$-variables. We generate 10 000 realizations of

$$\sum_{j=1}^{200}\sum_{l=1}^{200}(j\pi)^{-2}(l\pi)^{-2}\,\chi_{jl}^2(p)$$

to obtain the asymptotic critical values.

With $m := n - p$, Delgado's (1996) statistic is computed as

**Table 1.** Upper-tailed critical values of $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$

| $n$ | *Critical values for the following percentage points:* | | | | | |
|---|---|---|---|---|---|---|
| | *20%* | *10%* | *7.5%* | *5%* | *2.5%* | *1%* |
| $T_{\mathrm{GCM}}$ | | | | | | |
| 10 | 0.005233 | 0.006116 | 0.006494 | 0.007114 | 0.008090 | 0.009635 |
| 20 | 0.005456 | 0.006556 | 0.006928 | 0.007523 | 0.008541 | 0.010402 |
| 30 | 0.005484 | 0.006550 | 0.007029 | 0.007706 | 0.008650 | 0.010279 |
| 40 | 0.005543 | 0.006598 | 0.007006 | 0.007675 | 0.008679 | 0.010182 |
| 50 | 0.005558 | 0.006548 | 0.007012 | 0.007629 | 0.008623 | 0.010606 |
| 60 | 0.005542 | 0.006605 | 0.007064 | 0.007763 | 0.008797 | 0.010501 |
| 80 | 0.005556 | 0.006583 | 0.007044 | 0.007717 | 0.008755 | 0.010698 |
| 100 | 0.005539 | 0.006616 | 0.007080 | 0.007785 | 0.008697 | 0.010723 |
| $T_{\mathrm{GKS}}$ | | | | | | |
| 10 | 0.3300 | 0.3604 | 0.3629 | 0.3847 | 0.4032 | 0.4633 |
| 20 | 0.3708 | 0.4040 | 0.4168 | 0.4373 | 0.4659 | 0.4973 |
| 30 | 0.3851 | 0.4176 | 0.4310 | 0.4481 | 0.4774 | 0.5110 |
| 40 | 0.3951 | 0.4252 | 0.4374 | 0.4546 | 0.4786 | 0.5105 |
| 50 | 0.3975 | 0.4303 | 0.4414 | 0.4569 | 0.4826 | 0.5210 |
| 60 | 0.4022 | 0.4329 | 0.4446 | 0.4587 | 0.4823 | 0.5151 |
| 80 | 0.4076 | 0.4393 | 0.4516 | 0.4666 | 0.4932 | 0.5232 |
| 100 | 0.4088 | 0.4385 | 0.4515 | 0.4662 | 0.4921 | 0.5199 |

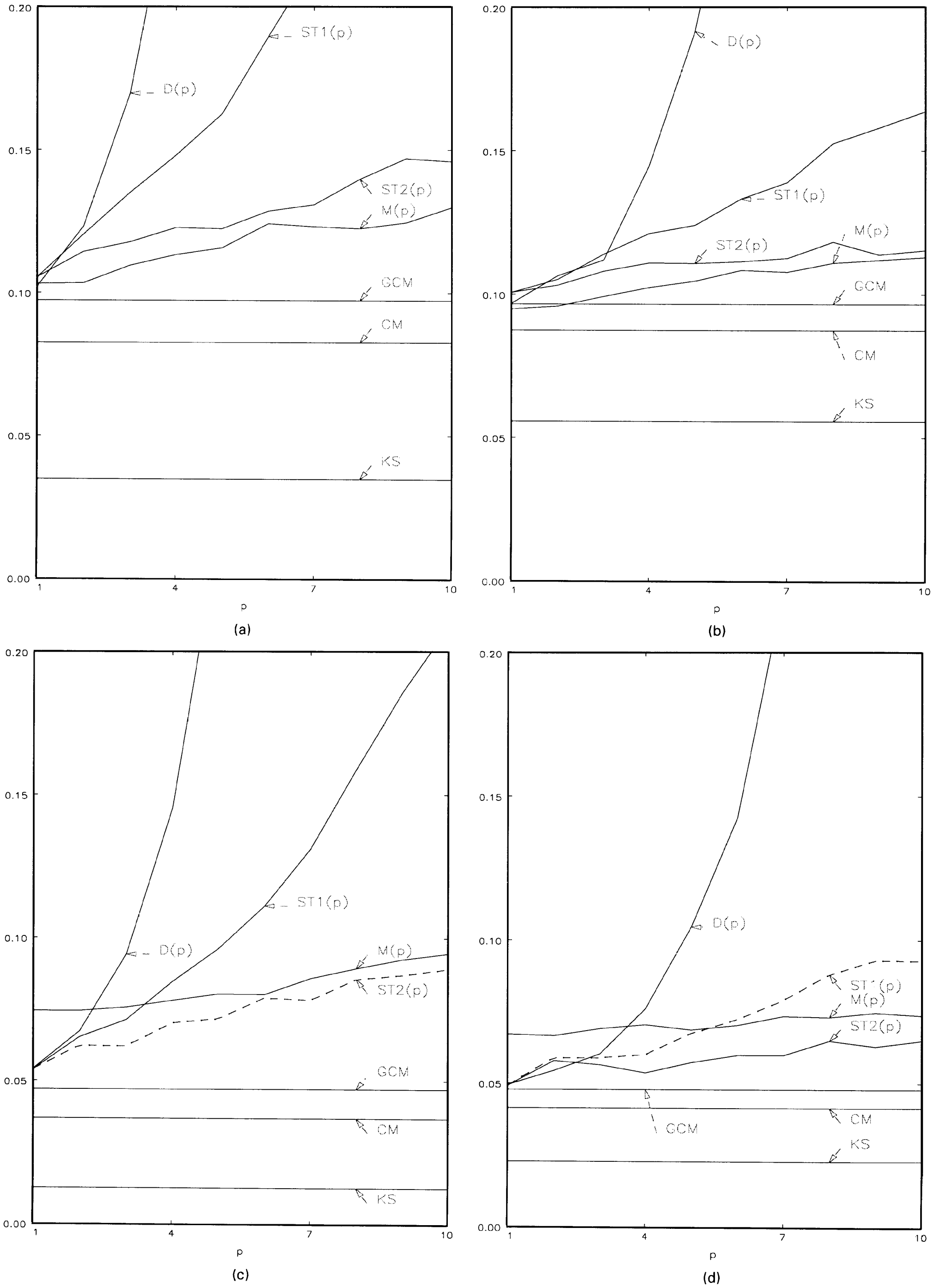


**Fig. 1.** Sizes by using asymptotics and the bootstrap (GCM, GKS, CM, KS, D($p$), M($p$), ST1($p$) and ST2($p$) denote the test statistics $T_{\mathrm{GCM}}$, $T_{\mathrm{GKS}}$, $T_{\mathrm{CM}}$, $T_{\mathrm{KS}}$, $T_{\mathrm{D}(p)}$, $T_{\mathrm{M}(p)}$, $T_{\mathrm{ST1}(p)}$ and $T_{\mathrm{ST2}(p)}$ respectively): (a) nominal level 10%, sample size $n = 40$; (b) nominal level 10%, sample size $n = 100$; (c) nominal level 5%, sample size $n = 40$; (d) nominal level 5%, sample size $n = 100$; (e) bootstrap level 10%, sample size $n = 40$; (f) bootstrap level 10%, sample size $n = 100$; (g) bootstrap level 5%, sample size $n = 40$; (h) bootstrap level 5%, sample size $n = 100$

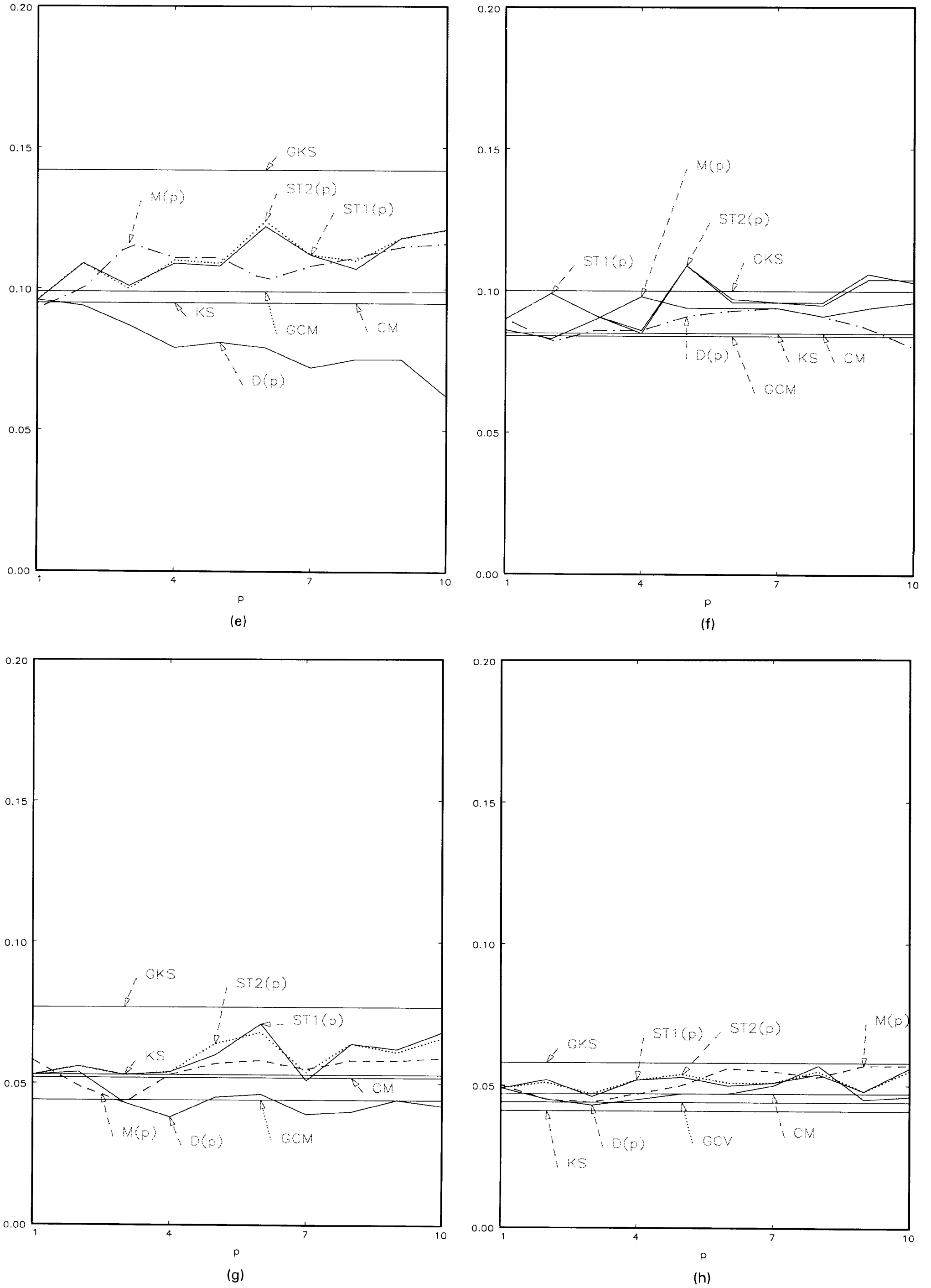


**Fig. 1** (*continued*)

$$T_{D(p)} := m \sum_{t=1}^{m} \left[ m^{-1} \sum_{s=p+1}^{n} \prod_{j=0}^{p} \mathbf{1}(X_{t-j} \leqslant X_{s-j}) - \prod_{j=0}^{p} \left\{ m^{-1} \sum_{s=p+1}^{n} \mathbf{1}(X_{t-j} \leqslant X_{s-j}) \right\} \right]^2.$$

Its limit distribution is non-standard (see Delgado (1996)). We approximate its asymptotic critical values using 10 000 realizations of the IID uniform random sample $\{X_t\}_{t=1}^{1000}$.

Finally, Hong's (1998) asymptotically $N(0, 1)$ test statistic is given by

$$T_{M(p)} := 90 \frac{\sum_{j=1}^{n-1} k^2(j/p)\{(n-j)\,\tilde{\sigma}^2(j) - 36^{-1}\}}{\left\{ 2 \sum_{j=1}^{n-2} k^4(j/p) \right\}^{1/2}},$$

with the Daniell kernel $k(z) = \sin(\pi z)/\pi z$, which is optimal over a class of kernels.

The statistics $T_{ST1(p)}$, $T_{ST2(p)}$, $T_{D(p)}$ and $T_{M(p)}$ involve the choice of lag order $p$. We consider $p$ in the range $1-10$ for each $n$. We study the size at the 10%, 5% and 1% levels, using 5000 realizations of an IID $N(0, 1)$ sample $\{X_t\}_{t=1}^{n}$, with $n = 20$, 40, 60, 100. To save space, Fig. 1 only reports the size at the 10% and 5% levels for $n = 40$ and $n = 100$. Because $T_{GCM}$, $T_{GKS}$, $T_{CM}$ and $T_{KS}$ do not depend on $p$, their horizontal lines do not mean that their sizes are a function of $p$. We first consider the size using the asymptotic theory (see Figs 1(a)–1(d)). The asymptotic critical values of $T_{GCM}$ are obtained from 10 000 realizations of expression (3.10) with truncation order $N = 100$. We do not include $T_{GKS}$ here, owing to the prohibitive tremendous time cost in obtaining its asymptotic critical values given my current computer resource. (However, we shall examine its size by using the bootstrap.) Of the seven tests, $T_{GCM}$ has the best sizes. $T_{ST1(p)}$, $T_{ST2(p)}$ and $T_{D(p)}$ also have accurate sizes for very small $p$. Given $n$, $T_{ST1(p)}$ and $T_{ST2(p)}$ show some overrejections as $p$ increases. $T_{D(p)}$ strongly overrejects, apparently because of the 'curse of dimensionality' caused by the joint dependence approach. $T_{M(p)}$ has some overrejection. The overrejections of $T_{ST1(p)}$, $T_{ST2(p)}$, $T_{D(p)}$ and $T_{M(p)}$ become weaker as $n$ increases. The tests $T_{CM}$ and $T_{KS}$ underreject, with $T_{CM}$ better than $T_{KS}$.

A bootstrap procedure is ideally suited to test hypothesis $\mathbb{H}_0$ and can be used as a remedy for an inadequate asymptotic approximation (see Skaug and Tjøstheim (1993b, 1996)). Figs 1(e)–1(h) report the bootstrap size using a procedure originally used in Hjellvik and Tjøstheim (1996) and Hong (1998). We generate 1000 realizations of the $N(0, 1)$ random sample $\{X_t\}_{t=1}^{n}$. For each realization of $\{X_t\}_{t=1}^{n}$ the bootstrap $p$-value of a test statistic is evaluated using a Gaussian kernel density estimator that is based on 100 bootstrap samples. Except for $T_{GKS}$ with $n = 40$, all the tests have reasonable bootstrap sizes. In particular, the bootstrap sizes of $T_{ST1(p)}$, $T_{ST2(p)}$, $T_{D(p)}$ and $T_{M(p)}$ are much better than their sizes using asymptotic critical values and are close to the nominal levels. The bootstrap size of $T_{GKS}$ is too large when $n = 40$ but becomes reasonable when $n = 100$. Interestingly, the bootstrap size of $T_{GCM}$ is not better than its size using asymptotic critical values in most cases.

### 5.2 Power

To investigate power, we consider the following alternatives: AR(1),

$$X_t = 0.3X_{t-1} + \epsilon_t;$$

ARFIMA(0, $d$, 0),

$$(1-B)^{0.3}X_t=\epsilon_t,$$
$$BX_t=X_{t-1};$$

bilinear,

$$X_t=\epsilon_t(0.2+0.5X_{t-1});$$

non-linear moving average,

$$X_t=\epsilon_{t-1}(0.8+\epsilon_t);$$

TAR(1),

$$X_t=\begin{cases}-0.5X_{t-1}+\epsilon_t & \text{if } X_{t-1}\leqslant 1,\\ 0.4X_{t-1}+\epsilon_t & \text{if } X_{t-1}>1;\end{cases}$$

EXP(1),

$$X_t=0.5X_{t-1}\ \exp(-0.5X_{t-1}^2)+\epsilon_t;$$

GARCH(1, 1),

$$X_t=\epsilon_t h_t^{1/2},$$
$$h_t=1+0.8h_{t-1}+0.19X_{t-1}^2;$$

LOGMAP,

$$X_t=4X_{t-1}(1-X_{t-1}),$$
$$X_0\sim U[0,\ 1].$$

Here $\{\epsilon_t\}$ is IID $N(0,\ 1)$, AR(1) and ARFIMA are first-order linear autoregressive and fractionally integrated processes, and the other alternatives are bilinear, non-linear moving average, threshold autoregressive, exponential autoregressive, generalized ARCH processes and an iterative logistic map. The logistic map behaves like a white noise process but is deterministic, exhibiting sensitive dependence on the initial value (e.g. Hall and Wolff (1995)). These are representative of commonly used time series models in practice.

Fig. 2 reports the size-corrected power at the 5% level for $n=100$, based on 1000 iterations for each alternative. We observe the following.

(a) $T_{\mathrm{GCM}}$ and $T_{\mathrm{GKS}}$ have less power than $T_{\mathrm{CM}}$ and $T_{\mathrm{KS}}$ against the AR(1) and ARFIMA models but $T_{\mathrm{GCM}}$ dominates $T_{\mathrm{CM}}$ and $T_{\mathrm{KS}}$ against the bilinear, non-linear moving average, TAR(1) and EXP(1) models. For the LOGMAP model, $T_{\mathrm{CM}}$ and $T_{\mathrm{KS}}$ have no power (as is expected); $T_{\mathrm{D}(p)}$ has decreasing power from 1 to 0 as $p$ increases, and all the other tests have unit power for all $p$.

(b) $T_{\mathrm{GCM}}$ dominates $T_{\mathrm{GKS}}$, and $T_{\mathrm{CM}}$ dominates $T_{\mathrm{KS}}$, for all the alternatives except LOGMAP.

(c) Except for the ARFIMA, GARCH and LOGMAP models, the powers of $T_{\mathrm{ST1}(p)}$, $T_{\mathrm{ST2}(p)}$, $T_{\mathrm{D}(p)}$ and $T_{\mathrm{M}(p)}$ achieve their own maxima when a correct lag order is used and then decline as $p$ increases, with the power of $T_{\mathrm{M}(p)}$ the least sensitive to $p$. Often, the power of $T_{\mathrm{GCM}}$ is lower than the maximal powers of $T_{\mathrm{ST1}(p)}$, $T_{\mathrm{ST2}(p)}$, $T_{\mathrm{D}(p)}$ and $T_{\mathrm{M}(p)}$, but by a small margin. In some cases (e.g. the bilinear, non-linear moving average and TAR(1) models), the power of $T_{\mathrm{GCM}}$ is the same as or even better than the maximal powers of $T_{\mathrm{ST1}(p)}$, $T_{\mathrm{ST2}(p)}$, $T_{\mathrm{D}(p)}$ and $T_{\mathrm{M}(p)}$ with a correct lag order.

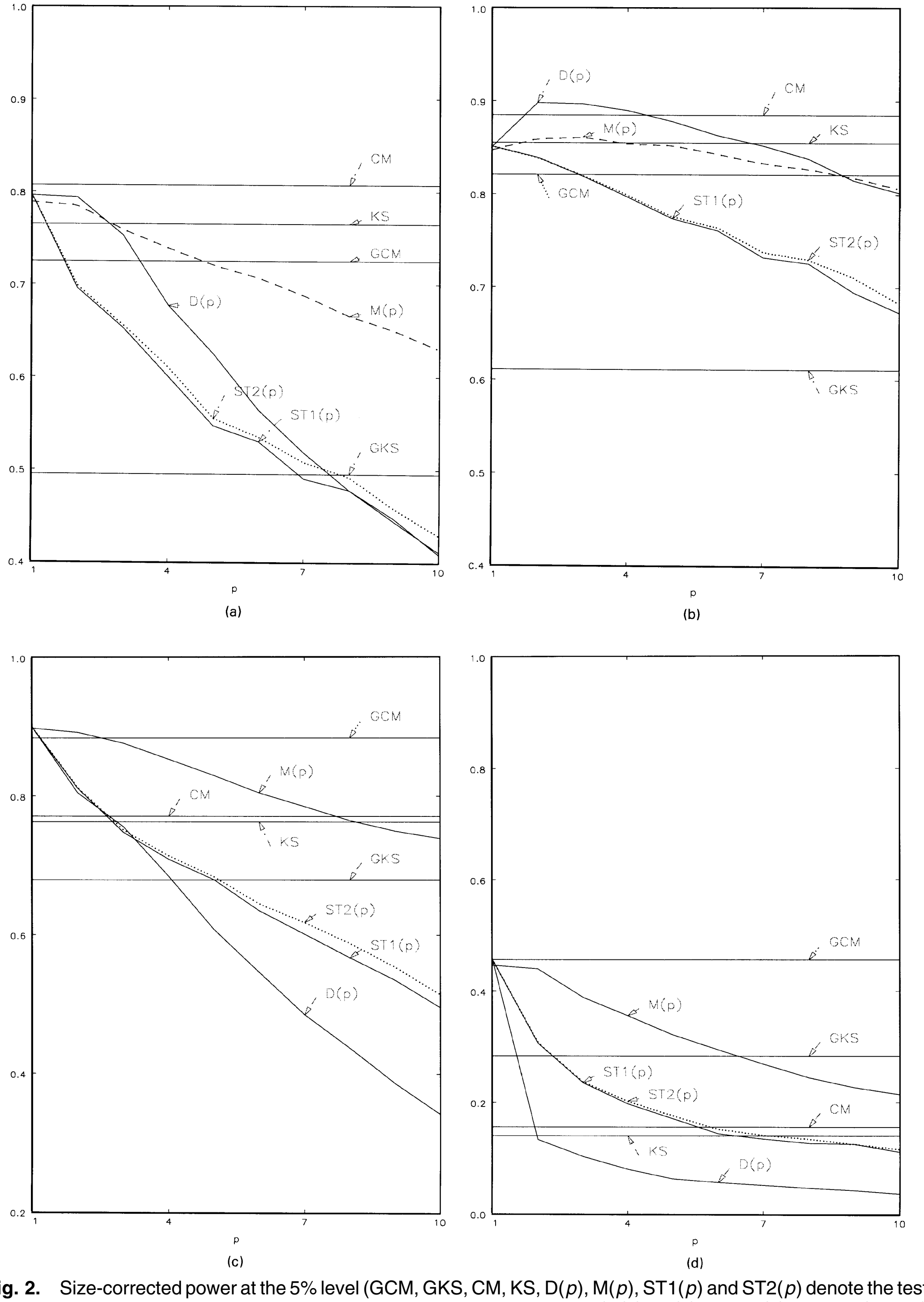


**Fig. 2.** Size-corrected power at the 5% level (GCM, GKS, CM, KS, D($p$), M($p$), ST1($p$) and ST2($p$) denote the test statistics $T_{\mathrm{GCM}}$, $T_{\mathrm{GKS}}$, $T_{\mathrm{CM}}$, $T_{\mathrm{KS}}$, $T_{\mathrm{D}(p)}$, $T_{\mathrm{M}(p)}$, $T_{\mathrm{ST1}(p)}$ and $T_{\mathrm{ST2}(p)}$ respectively): (a) AR(1) model; (b) ARFIMA(0, $d$, 0) model; (c) bilinear autoregressive model; (d) non-linear moving average model; (e) threshold AR(1) model; (f) exponential AR(1) model; (g) GARCH(1, 1) model; (h) logistic map

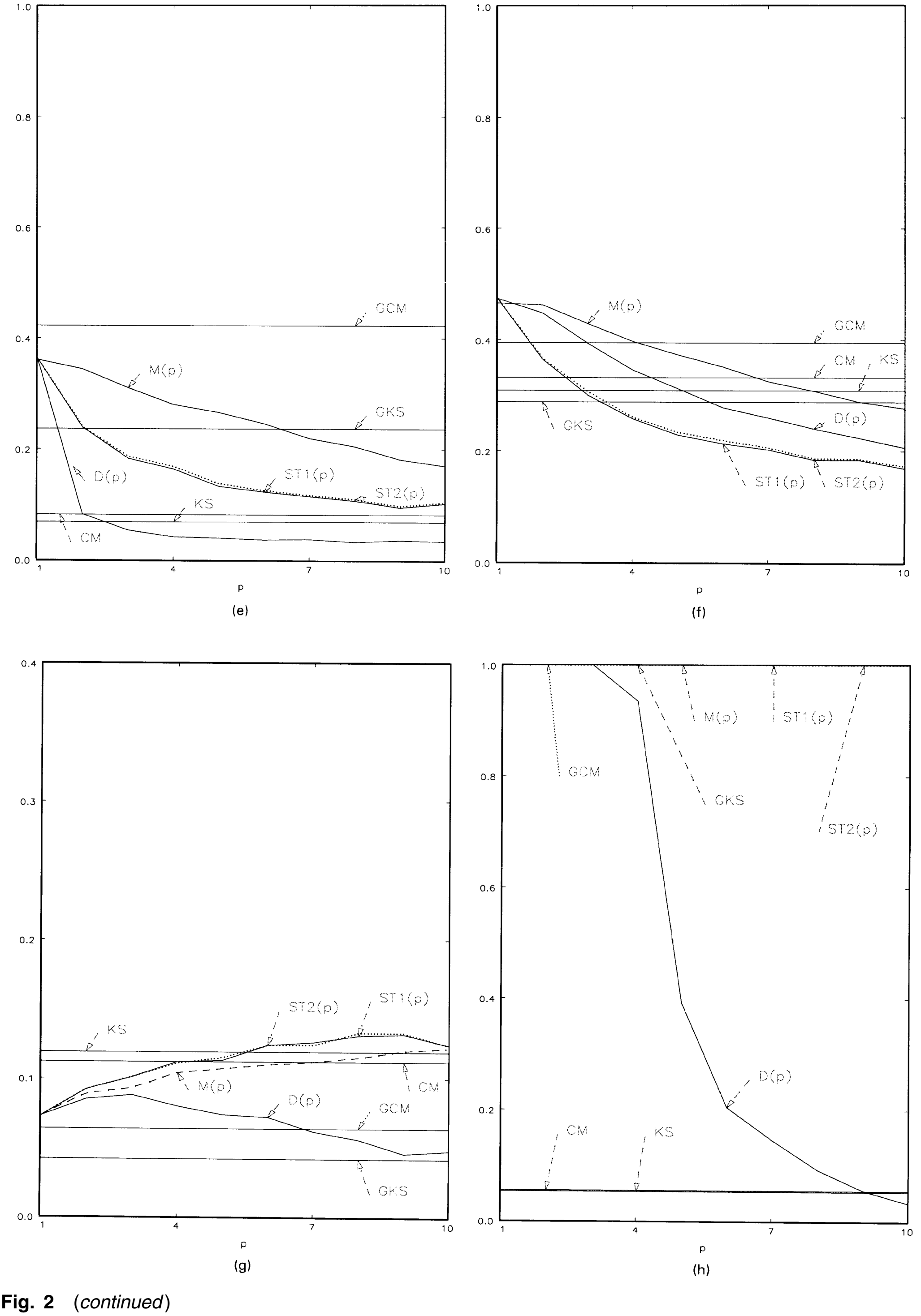


**Fig. 2** (*continued*)

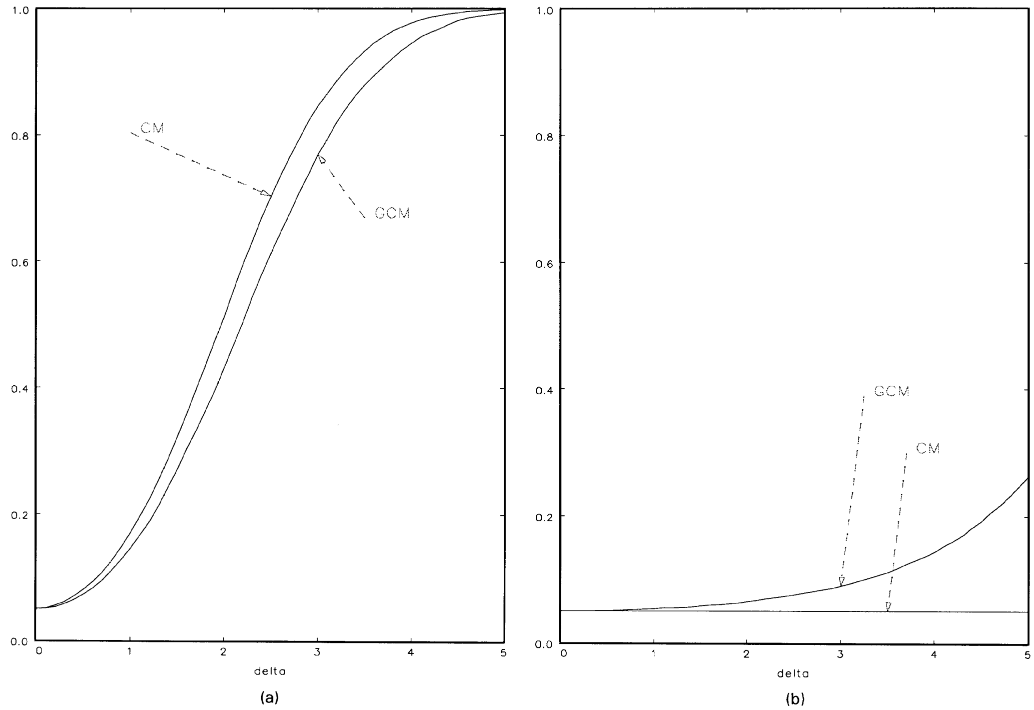


**Fig. 3.** Asymptotic local power of $T_{GCM}$ and $T_{CM}$ at the 5% level (GCM and CM denote the test statistics $T_{GCM}$ and $T_{CM}$): (a) local AR(1) model (5.1); (b) local ARCH(1) model (5.2)

(d) For the ARFIMA model where it is difficult to choose a proper lag order, the powers of $T_{ST1(p)}$ and $T_{ST2(p)}$ achieve their maxima at $p = 1$ and then decline as $p$ increases. The powers of $T_{D(p)}$ and $T_{M(p)}$ are unimodal, reaching their maxima at some $p > 1$.

(e) All the tests have low power against the GARCH model, with $T_{GKS}$ the least powerful. The powers of $T_{ST1(p)}$, $T_{ST2(p)}$ and $T_{M(p)}$ are slightly increasing with $p$, whereas the power of $T_{D(p)}$ is a unimodal function of $p$.

(f) $T_{D(p)}$ often has less power than the pairwise dependence tests $T_{ST1(p)}$, $T_{ST2(p)}$ and $T_{M(p)}$ for all the alternatives except ARFIMA. Its power also declines faster as $p$ increases. For the ARFIMA model, whose autocorrelation decays to 0 very slowly, however, $T_{D(p)}$ has better power than $T_{ST1(p)}$, $T_{ST2(p)}$ and $T_{M(p)}$ over a wide range of $p$.

To examine possible power loss of bootstrapping, we study the bootstrap power using 200 realizations of $\{X_t\}_{t=1}^n$. The results, not reported here, show that the bootstrap powers are very close to the size-corrected powers. We also consider the exponential distribution for $\epsilon_t$. The power patterns are largely similar to those under $N(0, 1)$ innovations, except that $T_{GCM}$ and $T_{GKS}$ become more powerful than $T_{CM}$ and $T_{KS}$ against the AR(1) model, and $T_{ST1(p)}$, $T_{ST2(p)}$ and $T_{M(p)}$ have gained some power and dominate the other tests in detecting the GARCH process.

The facts that EDF tests have good power against linear processes but poor power against ARCH processes in finite samples have also been documented in Skaug and Tjøstheim (1993a) and Hong (1998). To explain these stylized facts, we consider two local alternatives: AR(1),

$$g_n(X_t|\mathcal{F}_{t-1}) = \phi(X_t)\{1 + n^{-1/2}\delta X_t X_{t-1} + r_n(X_t,\, X_{t-1})\}, \tag{5.1}$$

and ARCH(1),

$$g_n(X_t|\mathcal{F}_{t-1}) = \phi(X_t)\{1 + n^{-1/2}\delta(X_t^2 - 1)(X_{t-1}^2 - 1) + r_n(X_t,\, X_{t-1})\}, \tag{5.2}$$

where $\phi(\cdot)$ is the $N(0,\, 1)$ density. Note that $\sqrt{n}\,\mathrm{corr}(X_t,\, X_{t-1}) \to \delta$ for equation (5.1) and $\sqrt{n}\,\mathrm{corr}(X_t^2,\, X_{t-1}^2) \to \delta$ for equation (5.2). The non-centrality process is

$$\mu(\gamma) = \delta\, \phi\{\Phi^{-1}(u)\}\, \phi\{\Phi^{-1}(v)\}$$

for equation (5.1) and

$$\mu(\gamma) = \delta\, \Phi^{-1}(u)\, \phi\{\Phi^{-1}(u)\}\, \Phi^{-1}(v)\, \phi\{\Phi^{-1}(v)\}$$

for equation (5.2), where $\Phi(\cdot)$ is the $N(0, 1)$ distribution function and $\Phi^{-1}(\cdot)$ its inverse function. Fig. 3 reports the asymptotic power of $T_{\mathrm{GCM}}$ and $T_{\mathrm{CM}}$ at the 5% level under models (5.1) and (5.2), as a function of $\delta$. The power of $T_{\mathrm{GCM}}$ is rather close to that of $T_{\mathrm{CM}}$ under model (5.1). Under model (5.2), however, $T_{\mathrm{GCM}}$ has low power over a wide range of $\delta$—its power increases with $\delta$, but very slowly. Thus, it is not surprising to have observed that $T_{\mathrm{GCM}}$ (and other EDF tests) has good power against linear processes but poor power against ARCH processes in finite samples. Intuitively, $T_{\mathrm{GCM}}$ has low power against ARCH processes because the distribution function, being the integral of the density function, smooths out changes in scale to a certain extent (the ARCH(1) process changes the scale of the conditional distribution of $X_t$ given $\mathcal{F}_{t-1}$). Because the density function is more sensitive to the change in scale, smoothed density-based tests are expected to have better power against ARCH processes. This is indeed the case for the smoothed tests of Skaug and Tjøstheim (1993b, 1996) and Hjellvik and Tjøstheim (1996).

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